3 Fourier Analysis

Fourier analysis is an extremely powerful technique and has a huge range of applications. There are entire textbooks devoted to it. I'll address only the very small part of it that applies most directly to the regression techniques discussed above. This application is often called **harmonic analysis**. You basically think of your signal as being a summation of many, many harmonic functions, each with a different temporal or spatial scale. The most familiar form is a sound or electromagnetic wave that is composed of many different frequencies which add up as sinusoids to give the full signal. The Fourier transform gives you coefficients which quantify how much each of the component harmonics contribute to the full signal.

For regression applications, I find it most instructive to think of harmonic analysis as simply the regression of your signal against each of the harmonics, and the Fourier coefficients as the regression and/or correlation coefficients. These coefficients then give you the variance explained by each frequency.

Say you have a temporal signal y with N samples, and you wish to regress it against the set of harmonic functions X which have the form:

$$cos(2\pi k\frac{t}{T})\,,\ k = 1, 2, 3, \ldots$$

and

$$sin(2\pi k\frac{t}{T})\,,\ k = 1, 2, 3, \ldots$$

Your signal may be expressed as the composition:

$$y = a_0 + \sum_{k=1}^{N/2} A_k cos(2\pi kt/T) + \sum_{k=1}^{N/2} B_k sin(2\pi kt/T) \tag{16}$$

where N is the number of samples, t is the time step between samples, T is the total time interval, and A_k B_k are the regression coefficients for each harmonic. Since the average of sines and cosines over the interval [0,T] is zero, $\overline{y} = a_0$.

For $k = 1$, the function completes 2π radians over the period [0,T]. This is the lowest resolvable frequency. For $k = N/2$, the function completes π radians in one time step. This is the highest resolvable frequency, also known as the **Nyquist frequency**. Each of the components of X have variance of $\overline{x_k'^2} = 0$, except for $A_{N/2}$ and $B_{N/2}$ which have variance of 1 and 0, respectively. Think of the sine wave at $k = N/2$: since it completes π radians in one time step, it has the same value at t as at t+dt, hence it's variance is zero. The cosine wave for $k = N/2$ has equal magnitude but opposite sign from t to t+dt, so it's variance is 1. I'll avoid getting into more detail here.

The **power spectrum** is the plot of components versus frequency, i.e., defining:

$$\begin{aligned} \frac{C_k^2}{2} &= \frac{A_k^2}{2} + \frac{B_k^2}{2}, \; k = 1 \ldots N/2 - 1 \\ C_{N/2}^2 &= A_{n/2}^2 \end{aligned}$$

the power spectrum is a plot of C_k^2 versus k.

The important thing is that this analysis also gives you the fraction of variance explained by each component:

$$r^2(y, k) = \frac{C_k^2}{2\sigma_y^2} \tag{17}$$

where σ_y^2 is the total variance of y.

And, since I typically deal with FFTs when I'm doing regression analysis, I convert this power to variance explained at each frequency. Note that I do not multiply the Nyquist by 2:

```
IDL> power[0:N/2-1] = 2.0*power[0:N/2-1]
IDL> bandwidth = 2.0*!DPI/N
IDL> variance = power/(bandwidth*Total(power))
```

Note that the sum of this *variance* (when multiplied by the bandwidth) should equal one, i.e., all of the variance should be explained, i.e., doing this:

```
IDL> print,Total(variance)*bandwidth
```

should give an answer of one.

The tricky part is determining what frequencies this power spectrum or variance spectrum corresponds to. All you have to remember is that the elements of the power spectrum array correspond to the following frequencies:

spectrum[0] =	the mean
spectrum[1] =	one cycle per total time range, T
spectrum[2] =	two cycles per total time range, T
spectrum[N/2] =	Nyquist frequency, N/2 cycles per total time range, T

Basically, all you have to do is construct a frequency array given the knowledge of the total span of your input sample and the physical units involved. Say your input data has N elements that spans a total time T. The power or variance spectrum will have $N/2 + 1$ elements with frequencies you can define by:

```
IDL> freq = FINDGEN(N/2)/T
```

And the units of *freq* are cycles per T. If the units of T are seconds, then the units of *freq* are Hz.