4.8 THE TRAPEZOIDAL RULE

The first of the Newton–Cotes formulas, based on approximating $f(x)$ on (x_0, x_1) by a straight line, is also called the *trapezoidal rule*. We have derived it by integrating $P_1(x_s)$, but the familiar and simple trapezoidal rule can also be considered to be an adaptation of the definition of the definite integral as a sum. To evaluate $\int_a^b f(x)\,dx$, we subdivide the interval from a to b into n subintervals, as in Fig. 4.5. The area under the curve in each subinterval is approximated by the trapezoid formed by replacing the curve by its secant line drawn between the endpoints of the curve. The integral is then approximated by the sum of all the trapezoidal areas. There is no necessity to make the subintervals equal in width, but our formula is simpler if this is done. Let h be the constant Δx. Since the area of a trapezoid is its average height times the base, for each subinterval,

$$\int_{x_i}^{x_{i+1}} f(x)\,dx \doteq \frac{f(x_i) + f(x_{i+1})}{2}(\Delta x) = \frac{h}{2}(f_i + f_{i+1}), \tag{4.42}$$

and for $[a, b]$ subdivided into subintervals of size h,

$$\int_a^b f(x)\,dx \doteq \sum_{i=1}^{n} \frac{h}{2}(f_i + f_{i+1}) = \frac{h}{2}(f_1 + f_2 + f_2 + f_3 + \cdots + f_n + f_{n+1});$$

$$\int_a^b f(x)\,dx = \frac{h}{2}(f_1 + 2f_2 + 2f_3 + \cdots + 2f_n + f_{n+1}). \tag{4.43}$$

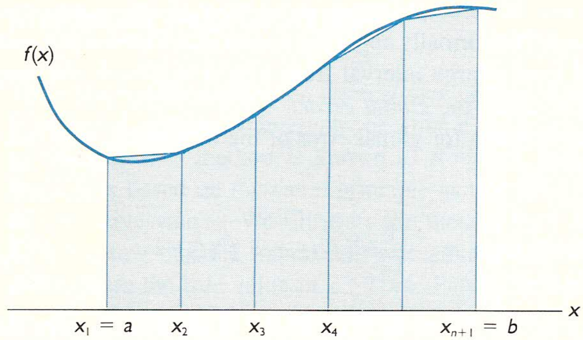

Figure 4.5

Equation (4.42) is identical to Eq. (4.36). Equation (4.43) is called the *extended trapezoidal rule*: it lets us apply the formula over an extended region where $f(x)$ is far from linear by applying the procedure to subintervals in which it can be approximated by linear segments. The formula is beautifully simple, and its applicability to unequally spaced values is useful in finding the integral of an experimentally determined function. It is obvious from Fig. 4.5 that the method is subject to large errors unless the subintervals are small, for replacing a curve by a straight line is hardly accurate.

EXAMPLE Suppose we wished to integrate the function tabulated in Table 4.3 over the interval from $x = 1.8$ to $x = 3.4$. The extended trapezoidal rule gives

$$\int_{1.8}^{3.4} f(x)\,dx \doteq \frac{0.2}{6}[6.050 + 2(7.389) + 2(9.025) + 2(11.023) + 2(13.464)$$
$$+ 2(16.445) + 2(20.086) + 2(24.533) + 29.964] = 23.9944.$$

The data in Table 4.3 are for $f(x) = e^x$, so the true value of the integral is $e^{3.4} - e^{1.8} = 23.9144$. We are off in the second decimal.

We have previously derived the error of the trapezoidal rule as Eq. (4.37). We repeat it here:

$$\text{Local error of trapezoidal rule} = -\frac{1}{12}h^3 f''(\xi_1), \qquad x_0 \le \xi_1 \le x_1.$$

Table 4.3

x	$f(x)$	x	$f(x)$
1.6	4.953	2.8	16.445
1.8	6.050	3.0	20.086
2.0	7.389	3.2	24.533
2.2	9.025	3.4	29.964
2.4	11.023	3.6	36.598
2.6	13.464	3.8	44.701

This error, it should be emphasized, is the error of only a single step, and is hence called the *local error*. We normally apply the trapezoidal formula to a series of subintervals to get the integral over a large interval from $x = a$ to $x = b$. We are interested in the total error, which is called the *global error*.

To develop the formula for global error of the trapezoidal rule, we note that it is the sum of the local errors:

$$\text{Global error} = -\frac{1}{12}h^3[f''(\xi_1) + f''(\xi_2) + \cdots + f''(\xi_n)]. \qquad \textbf{(4.44)}$$

In Eq. (4.44) each of the values of ξ_i is found in the n successive subintervals. If we assume that $f''(x)$ is continuous on (a, b), there is some value of x in (a, b), say $x = \xi$, at which the value of the sum in Eq. (4.44) is equal to $n \cdot f''(\xi)$. Since $nh = b - a$, the global error becomes

$$\begin{matrix}\text{Global error of} \\ \text{trapezoidal rule}\end{matrix} = -\frac{1}{12}h^3 nf''(\xi) = \frac{-(b-a)}{12}h^2 f''(\xi) = O(h^2). \qquad \textbf{(4.45)}$$

The fact that the global error is $O(h^2)$ while the local error is $O(h^3)$ is reasonable since, for example, if h is halved the number of subintervals is doubled, so we add together twice as many errors.

When the function $f(x)$ is known, Eq. (4.45) permits us to estimate the error of numerical integration by the trapezoidal rule. In applying this equation we bracket the error by calculating with the maximum and the minimum values of $f''(x)$ on the interval $[a, b]$.

For the example above, our error expression gives these estimates:

$$\begin{aligned}\text{Error} &= -\frac{1}{12}h^3 nf''(\xi), \qquad 1.8 \le \xi \le 3.4, \\ &= -\frac{1}{12}(0.2)^3(8)\begin{Bmatrix} e^{1.8} & \text{(min)} \\ e^{3.4} & \text{(max)}\end{Bmatrix} = \begin{Bmatrix} -0.0323 & \text{(min)} \\ -0.1598 & \text{(max)}\end{Bmatrix}.\end{aligned}$$

Alternatively,

$$\text{Error} = -\frac{1}{12}(0.2)^2(3.4 - 1.8)\begin{Bmatrix} e^{1.8} & \text{(min)} \\ e^{3.4} & \text{(max)}\end{Bmatrix} = \begin{Bmatrix} -0.0323 \\ -0.1598\end{Bmatrix}.$$

The actual error was -0.080. ■

If we did not know the form of the function for which we have tabulated values, we would estimate $h^2 f''(\xi)$ from the second differences.