Fourier Series (also called Harmonic Analysis)

A wave in the x direction may be decomposed into trigonometric components of the form:

\[ \psi(x) = \sum_{n=1}^{\infty} \left( A_n \sin k_n x + B_n \cos k_n x \right) \]

For a domain length of L where \( k = \frac{2\pi}{L} \) for n waves,

\[ \psi(x) = \sum_{n=1}^{\infty} \left( A_n \sin \frac{2\pi nx}{L} + B_n \cos \frac{2\pi nx}{L} \right) \]

The Fourier coefficients \( A_n \) and \( B_n \) are expressed as

\[ A_n = \frac{2}{L} \sum_{i=1}^{i_{\text{last}}} \psi_i(x) \sin \frac{2\pi nx}{L} \ dx \]

\[ B_n = \frac{2}{L} \sum_{i=1}^{i_{\text{last}}} \psi_i(x) \cos \frac{2\pi nx}{L} \ dx \]

where at \( i_{\text{last}} \), L is the length of the domain.

In calculus notation, this is written as

\[ A_n = \frac{2}{L} \int_{0}^{L} \psi(x) \sin \frac{2\pi nx}{L} \ dx \]

\[ B_n = \frac{2}{L} \int_{0}^{L} \psi(x) \cos \frac{2\pi nx}{L} \ dx \]

This allows one to retrieve \( A \) and \( B \) for each frequency!
The Fourier coefficient $C_n$ is the amplitude of the combined $A_n \sin \frac{2\pi n x}{L}$ and $B_n \sin \frac{2\pi n x}{L}$ terms for each $n$

$$C_n = \sqrt{A_n^2 + B_n^2}$$

with a phase shift $\phi_n$ computed by $\tan \phi_n = \frac{B_n}{A_n}$. The proof is below.

The signal power for each $n$ is

$$P_n = \frac{1}{2} C_n^2$$

The variance explained by each harmonic is

$$R_n^2 = \frac{i_{last} P_n}{(i_{last} - 1) s^2} \times 100\%$$

where $i_{last}$ is the last $i$ index at $L$ in the discretized domain.

$s^2$ is the sample variance

$$s^2 = \frac{1}{i_{last} - 1} \sum_{i=1}^{i_{last}} (Y_i - \overline{Y})^2$$

The ratio of $\frac{i_{last}}{i_{last} - 1}$ is a statistical artifact compensating for the fact this is a limited data sample, not a statistical population. This correction factor is ignored in most books.
Proof for $C_n$

If $\tan \phi = \frac{B}{A}$, it can be shown from Pythagorean theorem:

$$\sin \phi = \frac{B}{\sqrt{A^2 + B^2}} \quad \text{and} \quad \cos \phi = \frac{A}{\sqrt{A^2 + B^2}}$$

Consider:

$$\Upsilon(x) = A \sin x + B \cos x$$

$$= \sqrt{A^2 + B^2} \left[ \frac{A}{\sqrt{A^2 + B^2}} \sin x + \frac{B}{\sqrt{A^2 + B^2}} \cos x \right]$$

$$= \sqrt{A^2 + B^2} \left[ \sin x \cos \phi + \cos x \sin \phi \right]$$

C

trig identity!

$$= \sin (x + \phi)$$

Hence, $\Upsilon(x)$ can be rewritten:

$$\Upsilon(x) = C \sin (x + \phi)$$